Coulomb dissociation as an indirect method in Nuclear Astrophysics

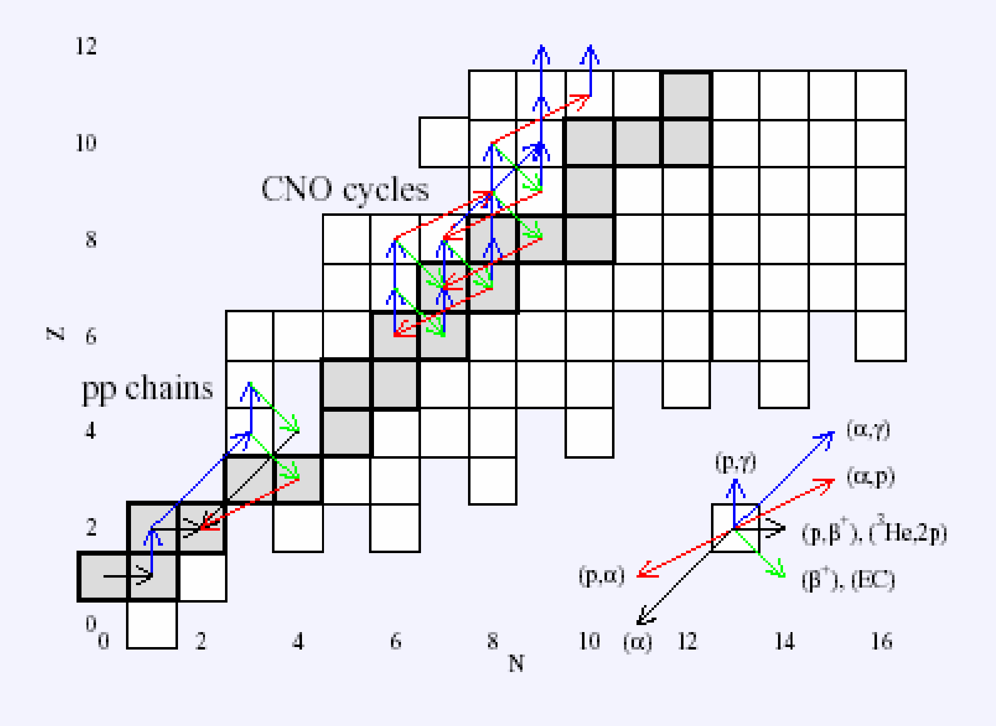
R. Chatterjee

Physique Nucléaire Théorique et Physique Mathématique
Université Libre de Bruxelles

Plan of the talk

- Indirect Methods in Nuclear Astrophysics
- The Coulomb dissociation method
- The ^{8}Li(n,γ)^{9}Li radiative capture reaction
- Conclusions

Indirect Methods in Nuclear Astrophysics. Why do we need them?



Nuclear Astrophysics :

Nuclear reaction rates at small energies are needed in many astrophysical models (stellar nucleosynthesis, novae, supernovae ….) for various processes (pp-chains, CNO cycle, r, p, s, rp, ..)

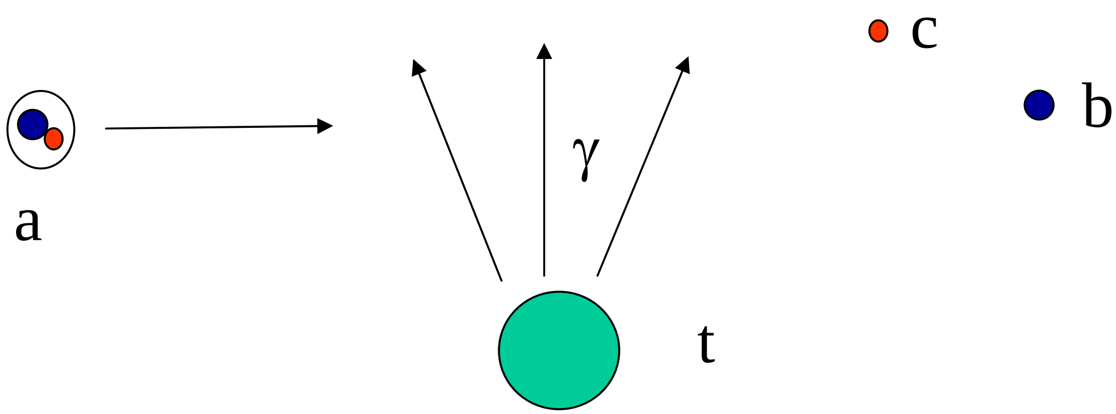
•Direct measurements are preferable, but are often difficult.

•Cross sections are small, unstable nuclei are involved, low yields...

•Alternative indirect methods, depending on the type of reaction, offers a way forward.

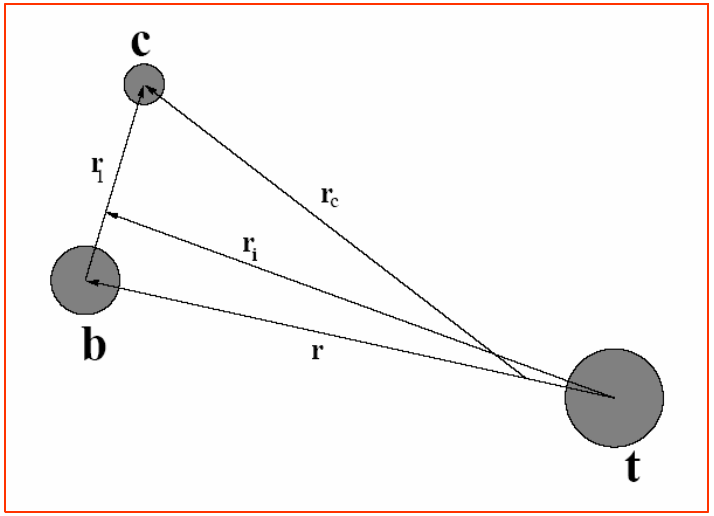
•**Coulomb Dissociation Method**, Asymptotic Normalization Coefficient, Trojan Horse Method.

Coulomb dissociation method - I



Projectile (**a**) breaks up into substructures (**b** and **c**) in the Coulomb field of target (**t**)

Theoretically : A fully quantum mechanical theory of Coulomb breakup reaction.



Three body final state : Jacobi coordinate system

Post form T – matrix

· ˜lxPh I #1 k »1 d »j k L, d1 »1 d »j -L, -dO˜ k-», ƒ dO/ W »1 d », W L, ƒ dx

Distorted Wave Born Approximation (DWBA)

/ W »1 d 7 J‰ ¤e », ƒ d1 ‰ »1 d »L‰ L, W d (Inelastic excitations of projectile are small)

Post form DWBA T – matrix

· I ? ¤e «@ #¤e «-@-O«@x#« k @ k «@O«‰ @‰ xW ¤ ´ C¤e »j k Lj -?L‰ d

Angular momentum couplings

Reduced T – matrix (six-dimensional integral)

C¤e »j k Lj -?L‰ d I #1 k »1 d »j k L, d#Wj -S, - O˜ k-», ƒ dO J‰ ¤e », ƒ d1 ‰ »1 d »L‰ L, W dx

Projectile structure information (bound state)

Finite Range DWBA (FRDWBA)

R.C., P. Banerjee, R. Shyam, NPA ***675*** *(2000) 477; NPA****692*** *(2001) 476*

Coulomb distorted waves

C¤e I ##W»8 j - 1 - ˘ dS,ƒ Ơ˜k-ƠJ¤e‰ »,ƒ dx#1 »1 d k »,W d#WF j - S,W Ơ1 »1 d ‰ »,W dx

Structure part

Dynamics part (expressed by bremsstralung integral analytically)

Adiabatic Model

P.Banerjee,J.A.Tostevin, I.J.Thompson Phys. Rev. C ***58*** *('98) 1337*

Assumption : Excitation energy of *b-c* system replaced by projectile binding energy.
Then three body wave function has the analytical form :

/ »1 ‰ dL| [»,ƒ L,W d I) ‰ »,ƒ d#W- L‰ S,ƒ 1 »1 ‰ d »L‰ L,d

Transition amplitude separates as :

C| ¤e [I ##W»1 - L‰ dS,ƒ Ơ˜k-ƠJ¤e‰ »,ƒ dx#1 »1 d k »j k L,d#WF j - S, Ơ1 »1 d ‰ »L‰ L,dx

The triple differential cross-section for Coulomb breakup :

q‾5 / q† kq, kq, –   I   QD / wX‰ G»† kL, kL, -d ƒ / Q« 1 ƒ   S ¤e   Œ¤e Ó L

The three body phase space factor

G»† kL, kL, -d   I   ¡ 1 e e ke -e P˘ k˘- / e P 1 e - 1 e - (j - S»L‰ 1 j k d / jº)

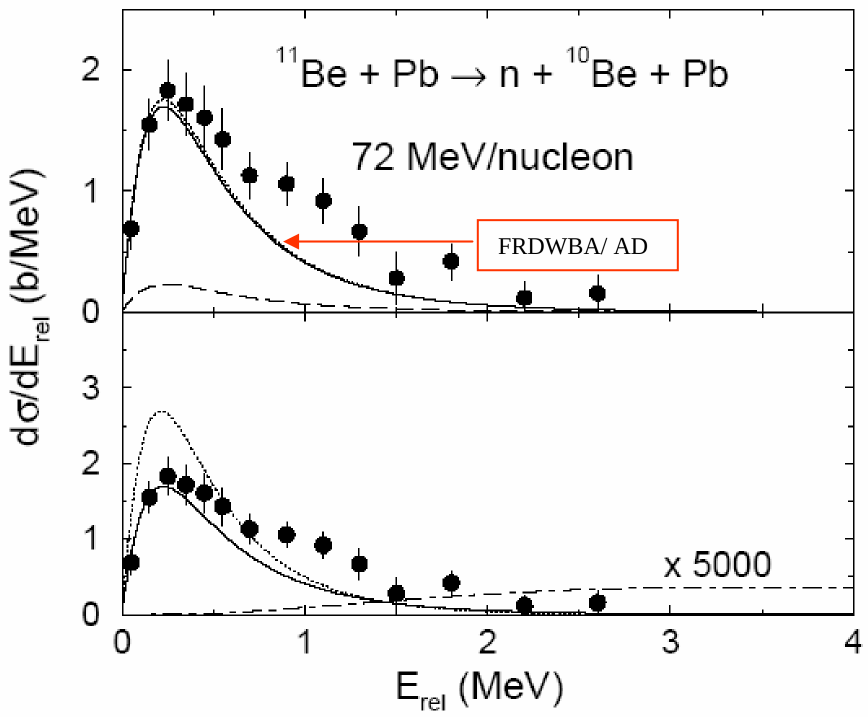
*H.Fuchs, Nucl. Instrum. Methods **200** ('82) 361*

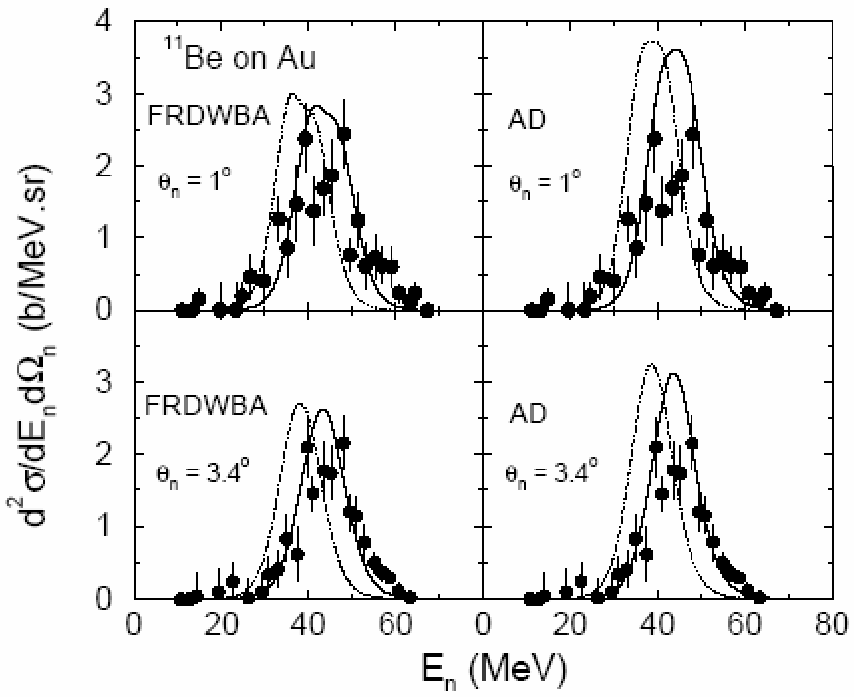
Comparison between FRDWBA and Adiabatic model

$^{10}\mathrm{Be}(0^+) \otimes 1s_{1/2}\nu$ SE = 0.504 MeV $C^2S = 0.74$
$^{10}\mathrm{Be}(2^+) \otimes 0d_{5/2}\nu$ SE = 3.872 MeV $C^2S = 0.17$

Relative energy spectra

Data: T.Nakamura et al. PLB ***331*** *('94) 296*



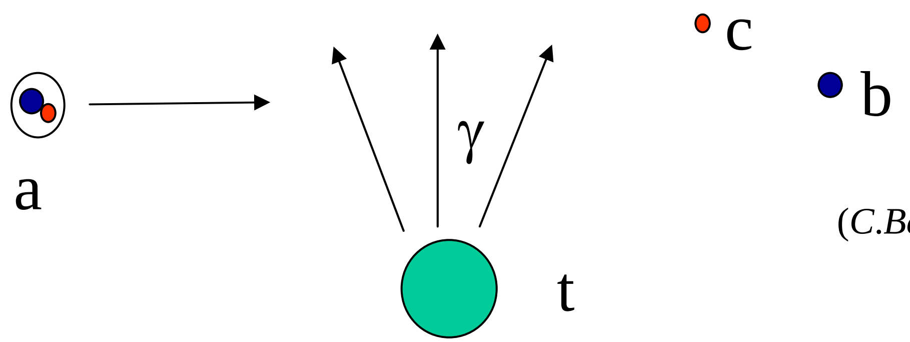


Neutron energy distribution

Data: R. Anne et al. Nucl. Phys. A ***575*** *('94) 123*

Coulomb dissociation method - II

Projectile **a** breaks up into substructures **b** and **c** in the Coulomb field of the target **t** (**t** is the source of photons **γ**)



(*C.Bertulani, G.Baur, H.Rebel, Nucl. Phys. A* ***458*** *(1986) 188*)

- Get Coulomb dissociation cross section : σ_{coul} $\qquad a + t \longrightarrow b + c + t$
- Relate with photo-disintegration cross section : σ_{phot} $\qquad a + \gamma \longrightarrow b + c$
- Relate to radiative capture cross section : σ_{rad} $\qquad b + c \longrightarrow a + \gamma$

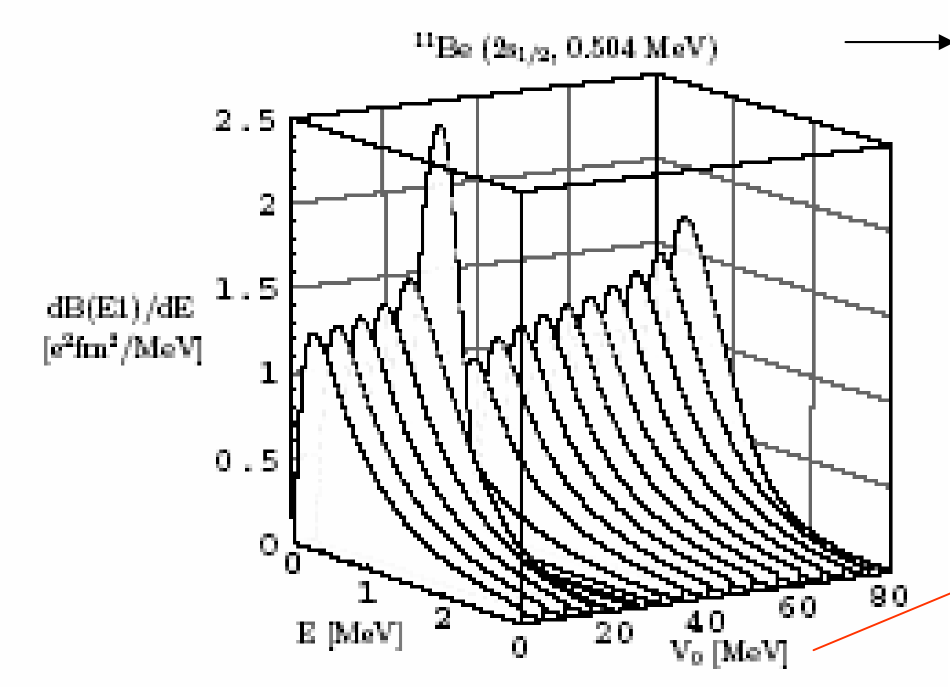
5 ¸‰q I Q»Q«‰ 1 ƒd / »Q«k 1 ƒd»Q«- 1 ƒd L8Q / LkQ- 5˘ ¡ 1P

j_i , i = a,b,c are spins of the particles
k_i are wave numbers

Experimentally how does one ensure this :
Heavy target, Forward angle scattering ensures dominance of Coulomb breakup

So what is the advantage of using our Coulomb dissociation theory ?

- We need only the ground state wave function of the projectile
- Do not need the position of the continuum states
- Our method is free from multipole strength distributions occurring in some formalisms



p - wave continuum

^{11}Be + t $\longrightarrow$ ^{10}Be + n + t

S. Typel and G.Baur NPA **759** 247 (2005)

Various depths of the potentials used to construct the *p* – wave continuum

Why is the $^{8}Li(n,\gamma)^{9}Li$ radiative capture reaction important?

Nucleosynthesis :

$^{1}H(n,\gamma)^{2}H(n,\gamma)^{3}H(d,n)^{4}He(^{3}H,\gamma)^{7}Li(n,\gamma)^{8}Li$

$^{8}Li(\alpha,n)^{11}B(n,\gamma)^{12}B$ …. (higher masses)

Competition between these reaction pathways

$^{8}Li(n,\gamma)^{9}Li(\beta^{-},\nu)^{9}Be(p,\alpha)^{6}Li$ (turns reaction flow back)

Larger $^{8}Li(n,\gamma)^{9}Li$ c.s. could lead to almost **50%** reduction in abundance of A $\geq$ 12 isotopes

R.A. Malaney and W.A. Fowler, *The Origin and Distribution of the Elements*, World Scientific (1988) p.76

Present Status

- ^{8}Li(n,γ)^{9}Li c.s. still uncertain

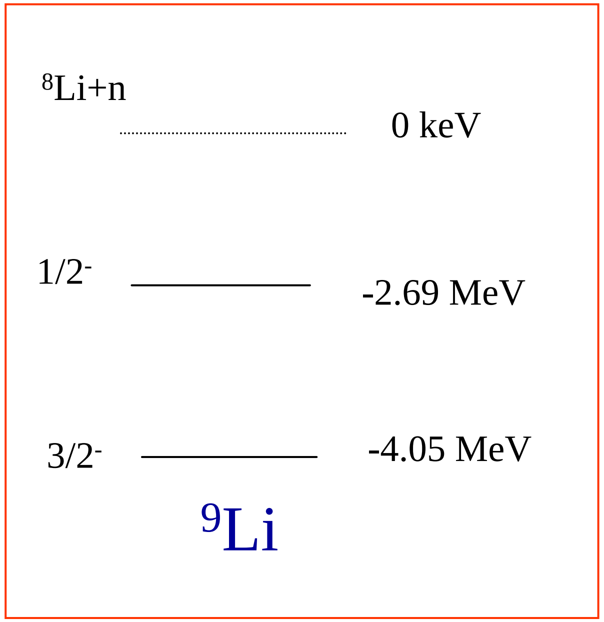
- Direct measurements are very difficult in this case
  : no *n* target and half life of ^{8}Li is too short (838ms)

Way forward : Use indirect measurements

Find the Coulomb dissociation c.s. of

$$^{9}\mathrm{Li} \longrightarrow {}^{8}\mathrm{Li} + \mathrm{n}$$

and relate back to the capture c.s.



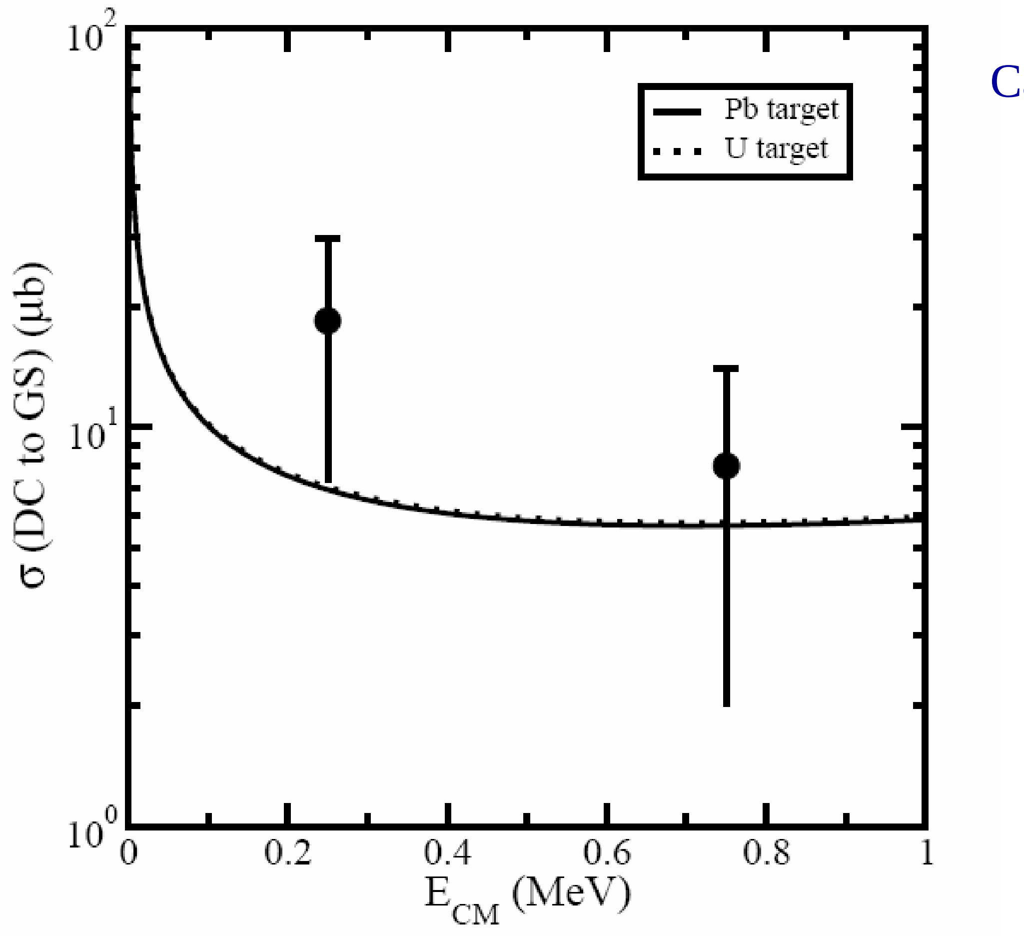
Experiments:
H. Kobayashi *et al*., Phys. Rev. C **67** (2003) 015806
P.D.Zecher *et al*., Phys. Rev. C **57** (1998) 959

Calculations:
Z. Mao, A. Champagne, Nucl. Phys. **A522** (1991) 568
P. Descouvemont, Astrophys. J. **405** (1993) 518
T. Rauscher et al., Astrophys. J. **429** (1994) 499
C.A. Bertulani, J. Phys. G **25** (1999) 1959

$^{8}Li(n,\gamma)^{9}Li$ cross section

Direct Capture (DC) to ground state (GS) of ^{9}Li

R.C., P. Banerjee, R. Shyam



Expt Data: *P.D.Zecher et al., PRC **57** (1998) 959*

Calculate Coulomb dissociation c.s of ^{9}Li:

$^{9}Li + t \longrightarrow {}^{8}Li + n + t$, t = Pb, U

Get : $n + {}^{8}Li \longrightarrow {}^{9}Li + \gamma$

Spectroscopic factor for the ground state of $^{9}Li = {}^{8}Li \times n$

- Shell Model calculations : 0.94
(Used by previous theoretical calculations)

- Experimentally determined from *transfer reactions* : 0.64 +/- 0.14
*[Z.H. Li et al. PRC **71**, 052801(R) (2005)]*

The ^{8}Li(n,γ)^{9}Li reaction rate at T_9 = 1

Reaction rate per particle pair :

(C.Rolfs & W.Rodney : Couldrons in the Cosmos)

#5XxI ‡ m / »LT · d D@ q ƒ | Q D @ H 5S† S#1 † | »LT · d q †

k_B = Boltzman constant, μ = reduced mass, T = stellar temperature (T_9 X 10^9 K)

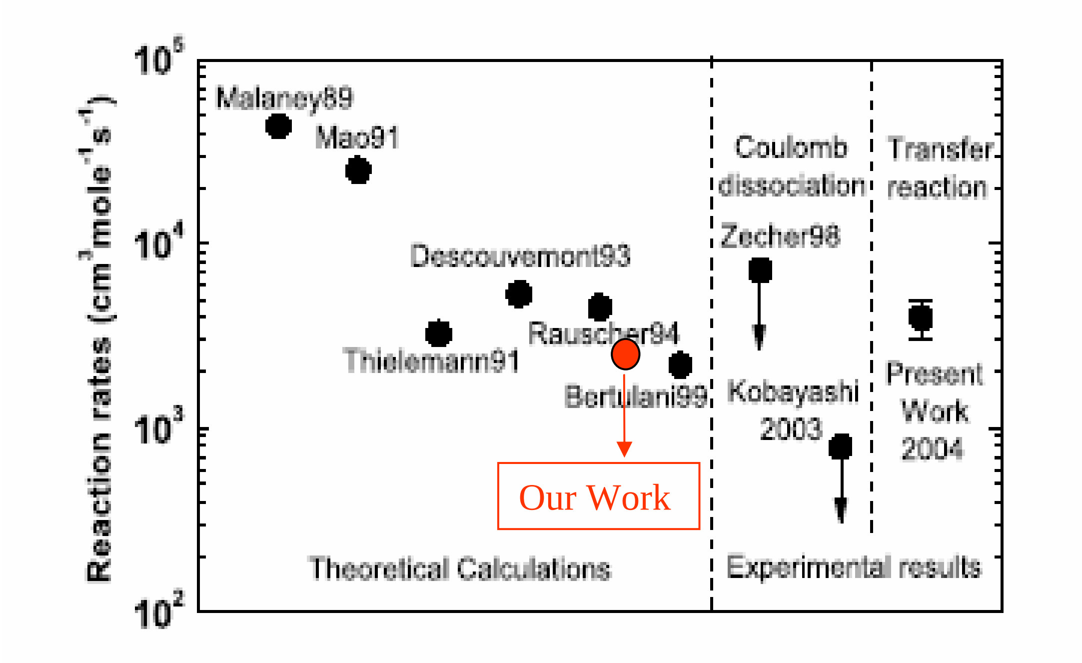
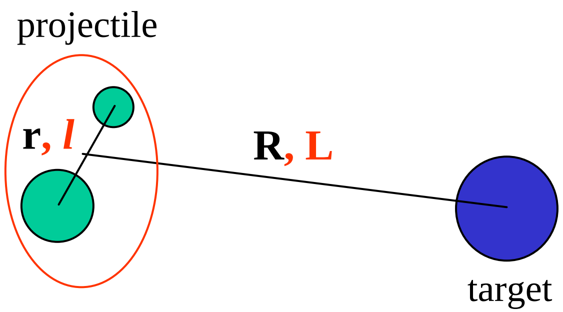


Fig. from *Z.H. Li et al. PRC* ***71****, 052801(R) (2005)*

The ^{8}Li(n,γ)^{9}Li reaction rate is not sufficiently large enough to affect the production of A > 12 elements.

Continuum Discretised Coupled Channels (CDCC) Method



Get bound and scattering states $\boldsymbol{u_{nl}(r)}$ (and eigenenergies $\boldsymbol{\varepsilon_{nl}}$) by the Lagrange mesh method

Three – body wavefunction

Defining channel $\boldsymbol{c = \{n,l,L\}}$ solve coupled channel equation using ***R***-matrix technique

Coupling potentials (arise from fragment – target interactions)

Get collision matrix, differential cross sections.

Conclusions and Perspectives

- Indirect methods in Nuclear Astrophysics, like the Coulomb dissociation method..., offers a way forward when the direct method is too difficult or is not feasible.

- The ^{8}Li(n,γ)^{9}Li reaction rate is not sufficiently large enough to affect the production of A > 12 elements.

- New inputs and constraints are required from Nuclear Physics theory and measurements to constrain models in Astrophysics.

- Plan to probe the effect of the continuum in reaction theories with the CDCC and Lagrange mesh method.

Conclusions and Perspectives

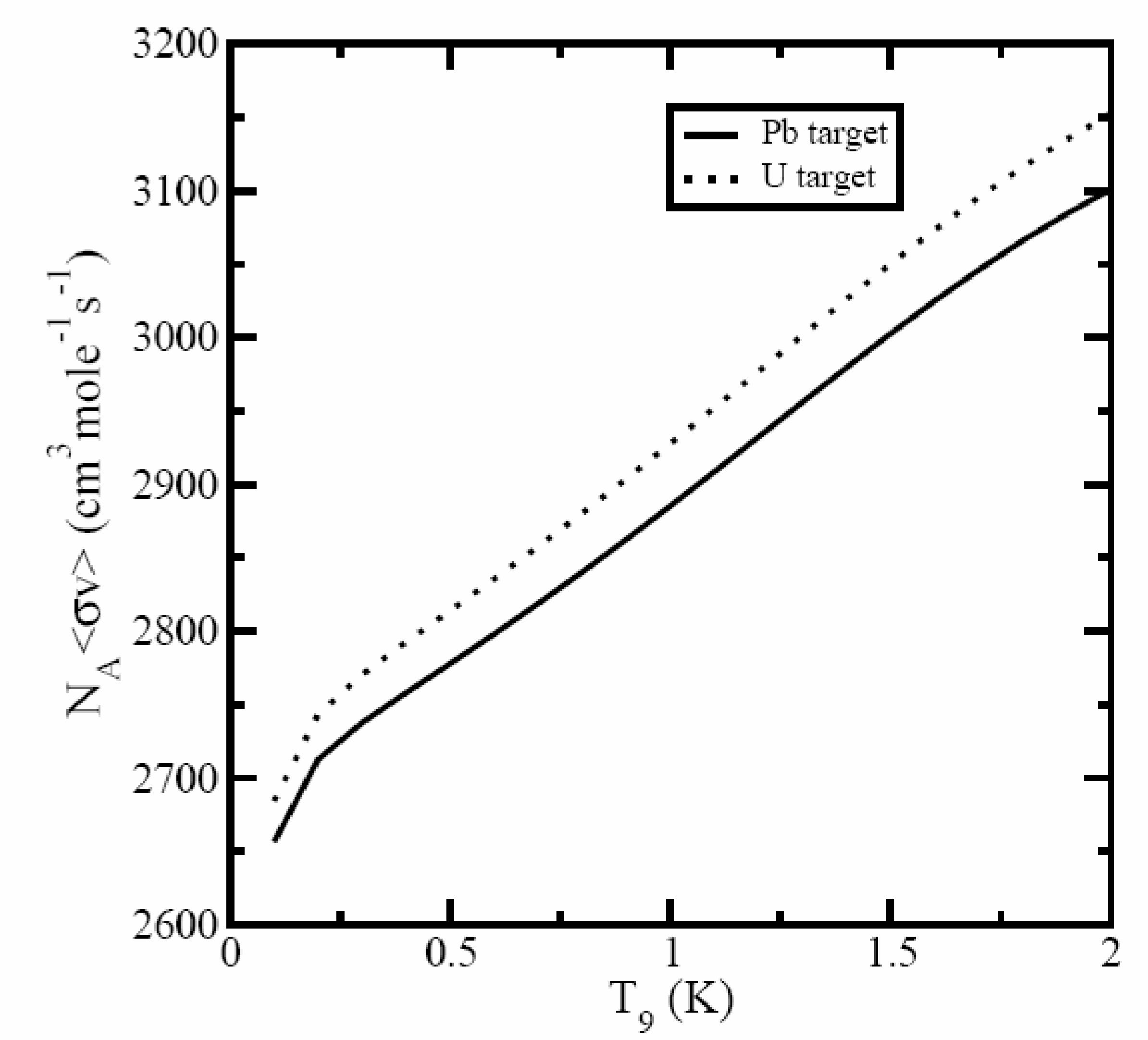
1. New inputs and constraints are required from Nuclear Physics theory and measurements to constrain models in Astrophysics.

2. Indirect methods in Nuclear Astrophysics, like the Coulomb dissociation method..., offers a way forward when the direct method is too difficult or is not feasible.

3. The ^{8}Li(n,γ)^{9}Li reaction rate is not sufficiently large enough to affect the production of $A > 12$ elements.

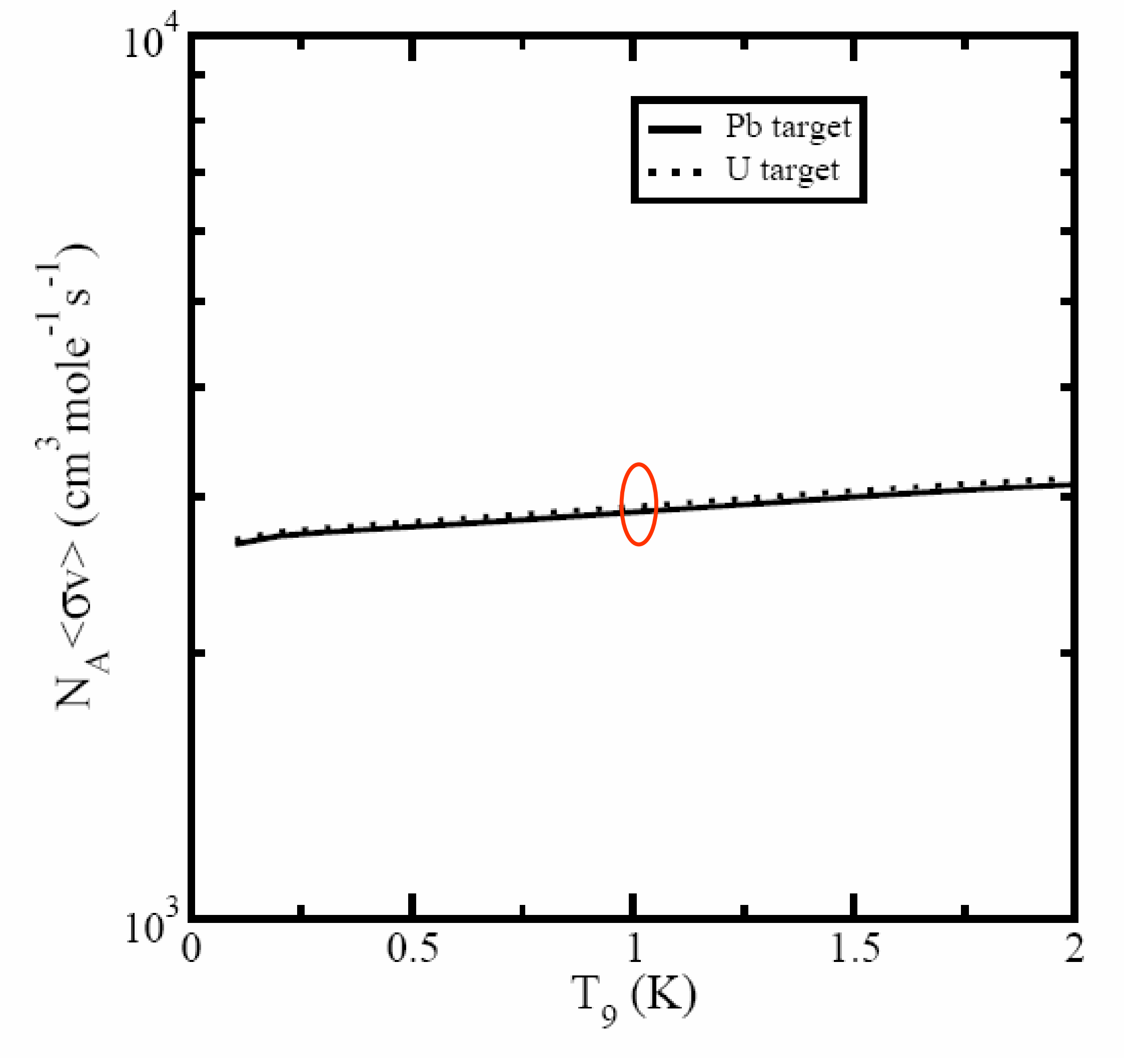
4. Plan to probe the effect of the continuum in reaction theories with the CDCC and Lagrange mesh method.

$^{8}Li(n,\gamma)^{9}Li$ reaction rate



Value at $T_9 = 1$ indicates that the other path with $^{8}Li(\alpha,n)^{11}B$ more likely to occur

$^{8}Li(n,\gamma)^{9}Li$ reaction rate



Value at T_9 = 1 indicates that the other path with $^{8}Li(\alpha,n)^{11}B$ more likely to occur

Breakup Reactions : To investigate the structure of exotic nuclei

Theoretically what do we need

• Final state interaction between breakup fragments and target nucleus taken to all orders. (Three or four body final states)

• Possible post acceleration of those fragments with higher charge to mass ratio.

• Full treatment of the breakup continuum (multistep processes) between the breakup channels, interference between different final state partial waves of core-halo relative motion.

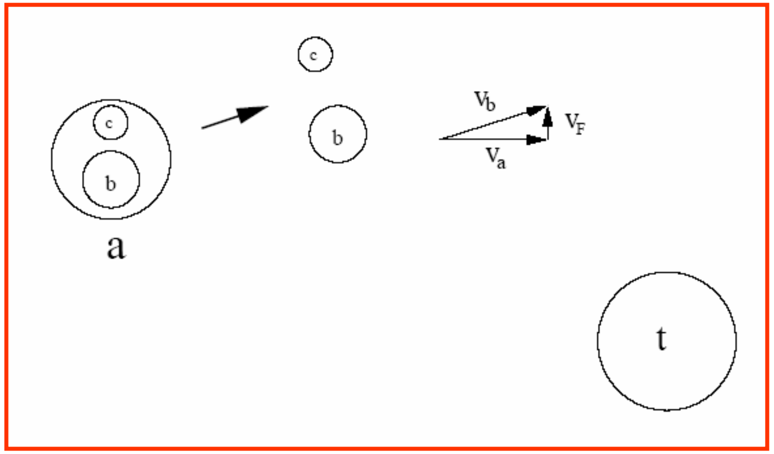
• Full Coulomb (*and nuclear*) interactions between the fragments and the target.

• Proper structure information of the projectile.
(inert) core + valence nucleon(s) : justifiable

• Dynamics of two-nucleon halo breakup.

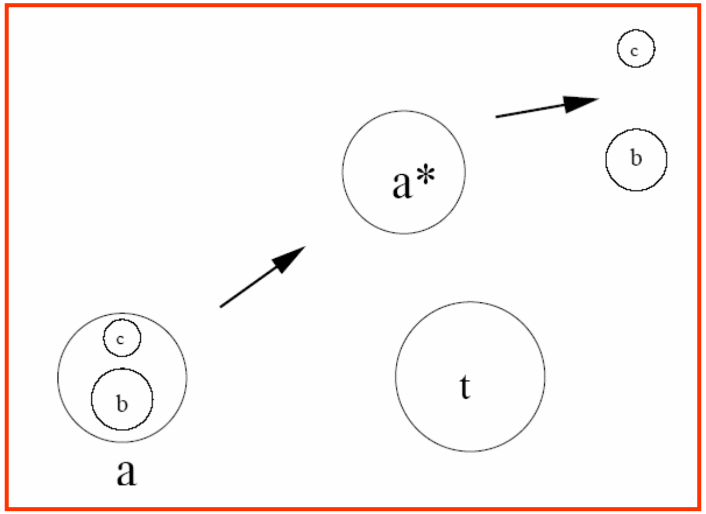
So far, no theory is complete in all respects.

Breakup Reactions



Direct Breakup : projectile (a) breaks into constituents (b, c) in the field of target (t)

Sequential Breakup : projectile (a) excited (a*) in the field of target (t), and then breaks up



The Hamiltonian

$$a + t \rightarrow b + c + t$$

Projectile (*a*) breaks up into fragments (*b* and *c*) in the field of target (*t*)

The Hamiltonian $H = \underbrace{T_b + T_c + T_t}_{\text{Kinetic energy terms}} + \underbrace{V_{bc} + V_{bt} + V_{ct}}_{\text{Two-body potentials}}$

Asymptotic hamiltonian in initial (*prior*) channel : $H_\alpha = T_b + T_c + T_t + V_{bc}$

initial (*prior*) channel interaction : $V_\alpha = H - H_\alpha = V_{bt} + V_{ct}$

Asymptotic hamiltonian in final (*post*) channel : $H_\beta = T_b + T_c + T_t$

final (*post*) channel interaction : $V_\beta = H - H_\beta = V_{bc} + V_{bt} + V_{ct}$

$$\boxed{H_\alpha + V_\alpha = H = H_\beta + V_\beta}$$

The Transition(T) - matrix

The T – matrix for the reaction $a + t \longrightarrow b + c + t$

· »l d˜ lxPh wW I ##Wj - S¸-P #Wj k S¸kP Ơ˜ k- l ˜ kP l ˜ -P Ơ/ »l d W x (Post form T – matrix)

exact three-body wave function

· »l d˜ ¸Wl ¸h wW I #/ »l d W Ơ˜ kP l ˜ -P G#Wj ‰ S¸‰ P J‰ »¸k-dx (Prior form T – matrix)

Introduce optical potentials (auxillary potentials), $\boldsymbol{U_{it}}$ $(i = b,c)$, to transform the plane waves into distorted waves (χ) with:

^· ¸-P l · ¸kP l j kP l j -Ph l »l d j- »¸-Pd l »l d jk »¸kPd I † l »l d j- »¸-Pd l »l d jk »¸kPd

^· ¸‰P l j ‰Ph l »l d j‰ »¸‰Pd I »† l V‰ d l »l d j‰ »¸‰Pd

binding energy of projectile

Using Gell-Mann Goldberger two potential formula :
M. Gell-Mann and M.L. Goldberger, Phys. Rev. 91 (1953) 398

· »l d˜ lxPh wW I #1 »l d j- », -p dl »l d jk », kP dO˜ k- l ~ kP l ~ -p l j kP l j -p O/ »l d W x

· »l d˜ , Wl , h wW I #/ »l d w O˜ -p l ~ kP l j ‰P Ol »l d j‰ », ‰P dJ‰ », k- dx

Distorted wave Born Approximation (DWBA)
G. R. Satchler, Direct Nuclear Reactions (Oxford Science)

/ »l d W 7 1 »l d j‰ », ‰P dJ‰ », k- d (Inelastic excitations of projectile are small)

· »l d˜ lxPh wW »[p T | d I #1 »l d jk », kP dl »l d j- », -p dO˜ k- Ol »l d j‰ », ‰P dJ‰ », k- dx

/ »l d w 7 1 »l d jk », kP dl »l d j- », -p d (V_{bc} is weak in final channel)

· »l d˜ , Wl , h wW »[p T | d I #1 »l d jk », kP dl »l d j- », -p dO˜ -p l ~ kP l j ‰P Ol »l d j‰ », ‰P dJ‰ », k- dx

· »l d˜ lxPh wW »[p T | d I · »l d˜ , Wl , h wW »[p T | d

The alternate Prior form T - matrix

/ »1 d 7 1 »1 d »‚‰PdJ »1 d ‰) Lj w »‚k-d (*V_{bc}* is important in final channel)

relative motion (b-c) wave function

center of mass motion

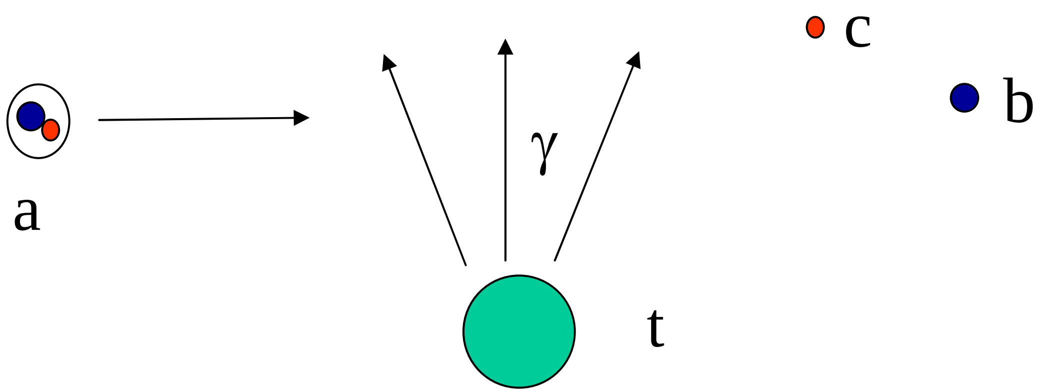
• »1 d‰·PL˘‚Wl‚h »[p T | dI #1 »1 d »‚‰PdJ »1 d ‰) Lj w »‚k-dO˘-p l ~ kPOl »1 d j‰ »‚‰PdJ‰ »‚k-dx

In the semi classical limit (neglecting nuclear effects) ⟶ Alder-Winther theory of Coulomb excitation

• »1 d‰·PL˘‚Wl‚h »[p T | dW • »1 d˜‚Wl‚h »[p T | d

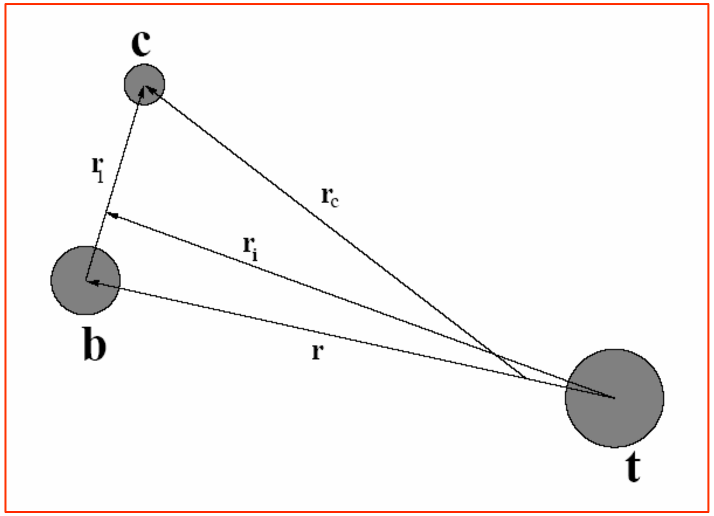
Coulomb Breakup of one-neutron halo nuclei

*(R.Chatterjee, P. Banerjee, R. Shyam, Nucl. Phys. A **675** (2000) 477)*



Projectile (**a**) breaks up into substructures (**b** and **c**) in the Coulomb field of target (**t**)

Theoretically : A fully quantum mechanical theory of Coulomb breakup reaction.



Three body final state : Jacobi coordinate system

Post form DWBA T – matrix

· I ? ¤e «@ #¤e «-@-O«@x#«$_k$@$_k$«@O«‰@‰xW¤´¤C¤e »j$_k$Lj-?L‰d

Angular momentum couplings

Reduced T – matrix (six-dimensional integral)

C¤e »j$_k$Lj-?L‰d I #1»1 d »j$_k$L,d#Wj-S,- Ó˜$_k$-»,ƒ dJ¤e‰ »,ƒ d1»1‰ d »L‰L,Wdx

Projectile structure information (bound state)

Zero Range (ZR) Approximation

G.R. Satchler, Direct Nuclear reactions, Oxford

~ $_k$-»,ƒ dJ¤e‰ »,ƒ d I [$_H$F»,ƒ d

Baur and Trautmann (BT)

G.Baur,W.Trautmann, Nucl Phys A ***199*** *('73)218*

,W 7 ,

Local Momentum Approximation (LMA)

R.Shyam, M.A.Nagarajan Ann. Phys. (NY) ***163*** *('85) 285*

Finite Range DWBA (FRDWBA)

1»1 d »j$_k$L,d I #1 - - S,ƒ 1»1 d »j$_k$L,Wd

Local Momentum : ˘ I tW ,W

- I e -|e ‰

FRDWBA

Coulomb distorted waves

C¤e I ##W»8j-1 - ˘ dS¸ƒ O˜k-QJ¤e‰ »¸ƒ dx#1k»1 d »¸W d#WFj-S¸W Q1‰»1 d »¸W dx

Structure part

Dynamics part (expressed by bremsstralung integral analytically)

Adiabatic Model

J.A.Tostevin, S.Rugmai, R.C.Johnson, Phys. Rev. C ***57*** *('98) 3225*

Assumption : Excitation energy of *b-c* system replaced by projectile binding energy. Then three body wave function has the analytical form :

/ ‰»1 dL| [»¸ƒ L¸W d I) ‰ »¸ƒ d#W- L‰ S¸ƒ 1‰»1 d »L‰ L¸d

Transition amplitude separates (with one less approximation than FRDWBA)

C¤e| [I ##W»1 - L‰ dS¸ƒ O˜k-QJ¤e‰ »¸ƒ dx#1k»1 d »j k L¸d#WFj-S¸ Q1‰»1 d »L‰ L¸dx

The triple differential cross-section for Coulomb breakup :

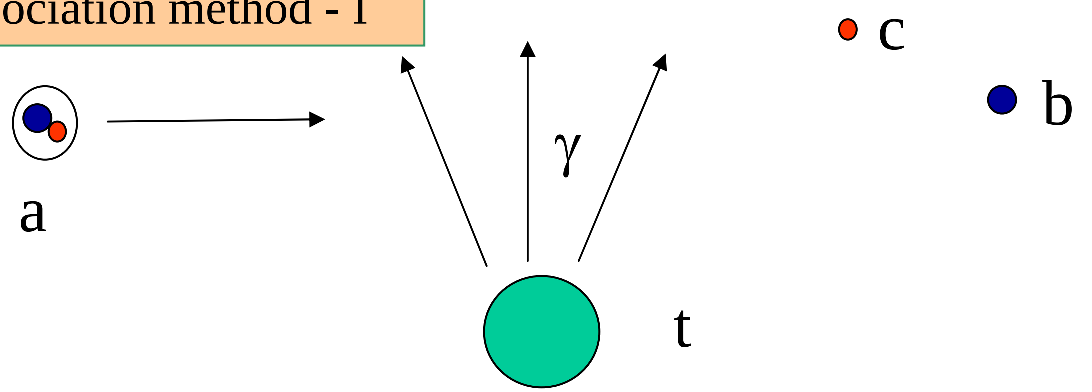
q¯5 / q† kq, kq, – I QD / wX‰ G»† kL, kL, –d ƒ / Q«‰ l ƒ S ¤e ŒC¤e ÓL

The three body phase space factor

G»† kL, kL, –d I ¡1 e e ke –e P˘ k˘– / e P l e – l e – j – S»L‰ l j k d / jº

H.Fuchs, Nucl. Instrum. Methods ***200*** *('82) 361*

Coulomb dissociation method - I



Projectile (**a**) breaks up into substructures (**b** and **c**) in the Coulomb field of target (**t**)

The post form transition amplitude in Distorted Wave Born Approximation (DWBA) :

. »l d˜ lxPh wW »[p T | dI #1 »l d jk »¸kPd1 »l d j- »¸-PdÕ˜ k-Ö1 »l d j‰ »¸‰PdJ‰ »¸k-dx

Can be factorised into **Structure** and
Dynamics part (expressed in terms of analytical integrals, if **c** is uncharged)

Theoretically : A fully quantum mechanical theory of Coulomb breakup reaction and very sensitive to ground state structure information of projectile.

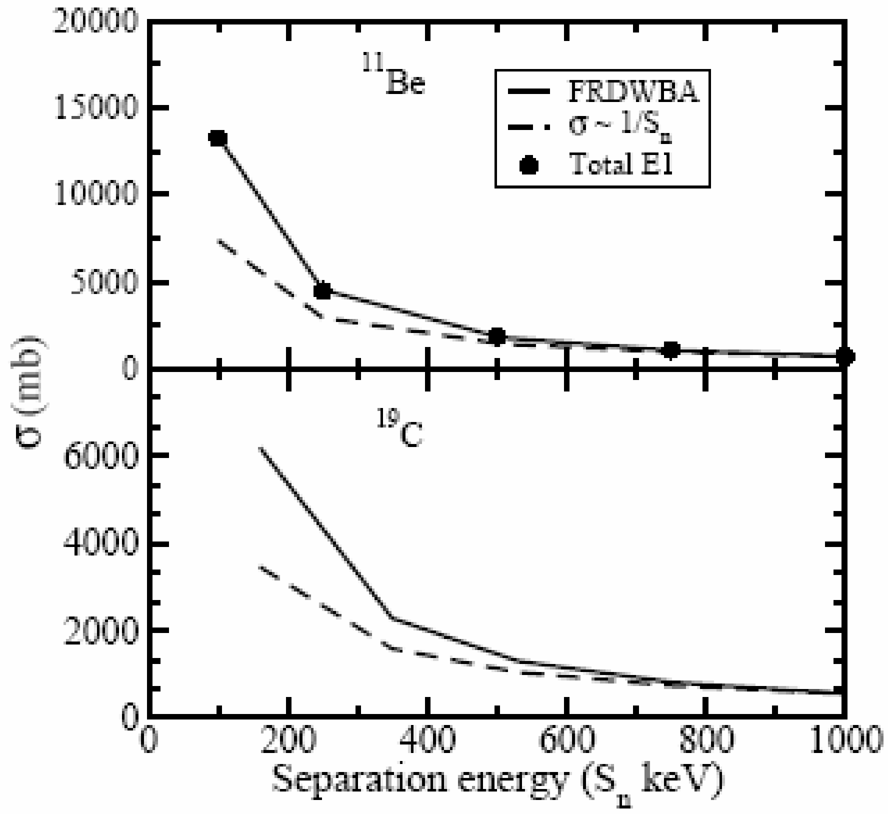
(Non Perturbative) **analytical expressions** for Coulomb breakup derived for the one(two)-neutron case with applications to **“exotic nuclei”**.

*(R.C., P. Banerjee, R. Shyam, Nucl. Phys. A **675** (2000) 477; Nucl. Phys. A **692** (2001) 476)*

Higher multipoles in the electromagnetic dissociation of halo nuclei

*R.C., L. Fortunato and A. Vitturi, EPJ **35** (2008) 213*

Total one neutron removal cross section vs. 'various' one-neutron separation energy (S_n) for ^{11}Be breakup on ^{208} Pb at 72 MeV/u



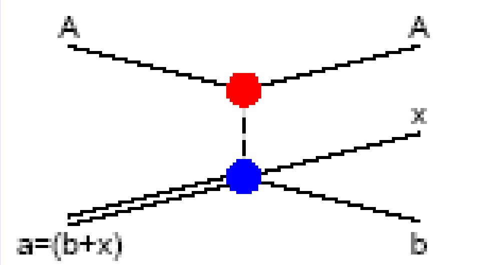
First order theory predicts :
*(M.A.Nagarajan, S.M.Lenzi, A.Vitturi EPJA **24** (2005) 63)*

Total breakup cross section ~ $1/S_n$
(for dipole transition)

Full quantal theory (FRDWBA) :
all multipoles included

Moving away from valley of stability towards the neutron drip line does not alter the predominance of dipole dissociation in the breakup process

Coulomb Dissociation II



$\Rightarrow$ cross section of Coulomb dissociation reaction

$$\frac{d^2\sigma}{dE_{bx}d\Omega_{Aa}} = \frac{1}{E_\gamma}\sum_{\pi\lambda}\sigma_{\pi\lambda}(a+\gamma\to b+x)\frac{dn_{\pi\lambda}}{d\Omega_{Aa}}$$

- virtual photon number $\frac{dn_{\pi\lambda}}{d\Omega_{Aa}}$ in quantal calculation or semiclassical approximation

  $\Rightarrow$ $E2$ enhancement $\frac{dn_{E2}}{d\Omega_{Aa}} \Big/ \frac{dn_{E1}}{d\Omega_{Aa}} \approx \frac{4\hbar^2 c^2}{E_\gamma^2 b^2}$, $M1$ suppression $\frac{dn_{M1}}{d\Omega_{Aa}} \Big/ \frac{dn_{E1}}{d\Omega_{Aa}} \approx \frac{v^2}{c^2}$

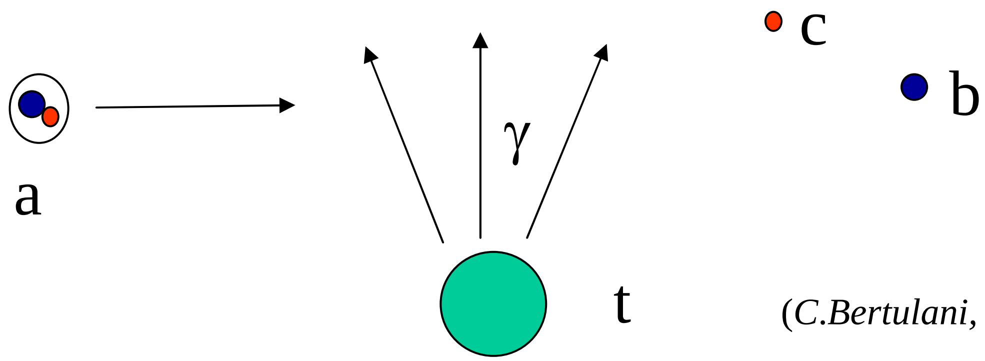
- theorem of detailed balance

$$\sigma_{\pi\lambda}(b+x\to a+\gamma) = \frac{2(2J_a+1)}{(2J_b+1)(2J_x+1)}\frac{k_\gamma^2}{k_{bx}^2}\sigma_{\pi\lambda}(a+\gamma\to b+x)$$

  with radiative capture and photo dissociation cross sections

- phase space factor $\frac{k_\gamma^2}{k_{bx}^2} = \frac{(E_{bx}+S_{bx})^2}{2\mu_{bx}c^2E_{bx}} \ll 1$ for (not too) small E_{bc}

Coulomb dissociation method - IIA



(*C.Bertulani, G.Baur, H.Rebel, Nucl. Phys. A* ***458*** *(1986) 188*)

Coulomb dissociation cross section :

q5 / q† k⁻ I ƒ / † 8 S D) 5D) ‰ l 8 , kl ¯d ¿D)

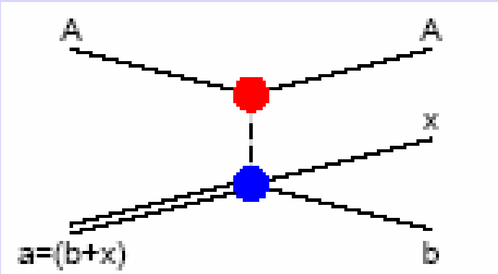
π = E/M λ = 1, 2, …

$\boldsymbol{\sigma_{\pi\lambda}}$: photodissociation cross section

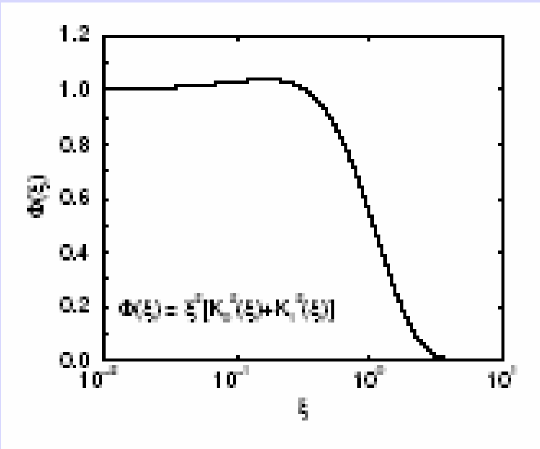
$\boldsymbol{n_{\pi\lambda}}$: equivalent photon number

$\boldsymbol{E_{\gamma} = E_{bc}}$ **+ *Q-value*** of the reaction

Coulomb Dissociation III



virtual photon spectrum (E1)



(Fermi 1924, Weizsäcker-Williams 1932)

characteristic parameters

- adiabaticity parameter

$\xi = \dfrac{\omega b}{\gamma v}$	$\hbar\omega$	excitation energy
	b	impact parameter
	v	projectile velocity

$\xi = 0$: sudden excitation

$\xi \gg 1$: adiabatic excitation

$\xi \approx 1 \Rightarrow E_{\rm exc}^{\rm max} \approx \gamma v \hbar / b$

- strength parameter

$\chi = \dfrac{Z_A e \langle f \| \mathcal{M}(\pi\lambda) \| i\rangle}{\hbar v b^{\lambda}}$	$Z_A e$	target charge
	$\mathcal{M}(\pi\lambda)$	multipole operator

χ small $\Rightarrow$ first-order perturbation theory sufficient

χ large $\Rightarrow$ higher-order effects

$$
\chi_b^{(-)*}(\mathbf{k}_b, \mathbf{r}_i) \;\; = \;\; e^{-\pi\eta_b/2}\Gamma(1+i\eta_b)e^{-i\mathbf{k}_b.\mathbf{r}_i}{}_1F_1(-i\eta_b, 1, i(k_b r_i + \mathbf{k}_b.\mathbf{r}_i)),
$$

$$
\chi_a^{(+)}(\mathbf{k}_a, \mathbf{r}_i) \;\; = \;\; e^{-\pi\eta_a/2}\Gamma(1+i\eta_a)e^{i\mathbf{k}_a.\mathbf{r}_i}{}_1F_1(-i\eta_a, 1, i(k_a r_i - \mathbf{k}_a.\mathbf{r}_i))
$$

$$
\frac{d^3\sigma}{dE_b d\Omega_b d\Omega_c} = \frac{2\pi}{\hbar v_a}\rho(E_b, \Omega_b, \Omega_c)\frac{4\pi^2\eta_a\eta_b}{(e^{2\pi\eta_b}-1)(e^{2\pi\eta_a}-1)}|I|^2 4\pi\sum_\ell |Z_\ell|^2.
$$

$$
\begin{aligned}
I \;\; &= \;\; -i\Big[B(0)\Big(\frac{dD}{dx}\Big)_{x=0}(-\eta_a\eta_b){}_2F_1(1-i\eta_a, 1-i\eta_b; 2; D(0)) \\
&+ \;\; \Big(\frac{dB}{dx}\Big)_{x=0}{}_2F_1(-i\eta_a, -i\eta_b; 1; D(0))\Big]
\end{aligned}
$$

where

$$B(x) = \frac{4\pi}{k^{2(i\eta_a + i\eta_b + 1)}} \Big[(k^2 - 2\mathbf{k}.\mathbf{k}_a - 2xk_a)^{i\eta_a} (k^2 - 2\mathbf{k}.\mathbf{k}_b - 2xk_b)^{i\eta_b} \Big], \tag{3.24}$$

$$D(x) = \frac{2k^2(k_a k_b + \mathbf{k}_a.\mathbf{k}_b) - 4(\mathbf{k}.\mathbf{k}_a + xk_a)(\mathbf{k}.\mathbf{k}_b + xk_b)}{(k^2 - 2\mathbf{k}.\mathbf{k}_a - 2xk_a)(k^2 - 2\mathbf{k}.\mathbf{k}_b - 2xk_b)} \tag{3.25}$$

with $\mathbf{k} = \mathbf{k}_a - \mathbf{k}_b - \delta\mathbf{k}_c$. Z_ℓ contains the projectile structure information and is given by

$$Z_\ell = \int dr_1 r_1^2 j_\ell(k_1 r_1) V_{bc}(\mathbf{r}_1) u_\ell(r_1), \tag{3.26}$$

with $k_1 = |\gamma\mathbf{k}_c - \alpha\mathbf{K}|$.